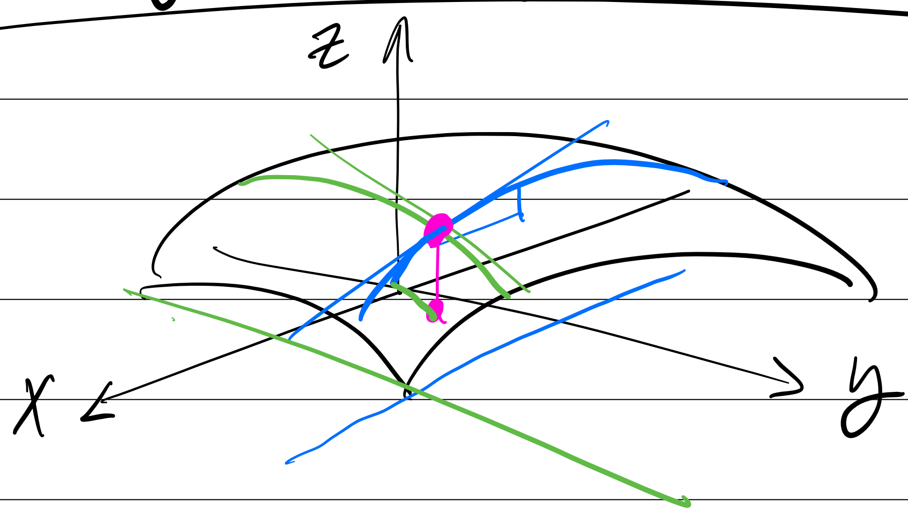
2.4 - Exact Equations

A partial derivative is an ordinary derivative holding all but one variable constant.

Notation: $f(x,y) = x^2y^3$

$f_x = \frac{\partial f}{\partial x} = 2xy^3$ $\qquad$ $\frac{\partial f}{\partial y} = 3x^2y^2 = f_y$

$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$ $\qquad$ $\frac{\partial^2 f}{\partial x \partial y} = 6xy^2 = f_{yx}$



$y = f(x)$

Calc 1:

$f'(x) = \frac{dy}{dx}$

$dy = f'(x)dx$

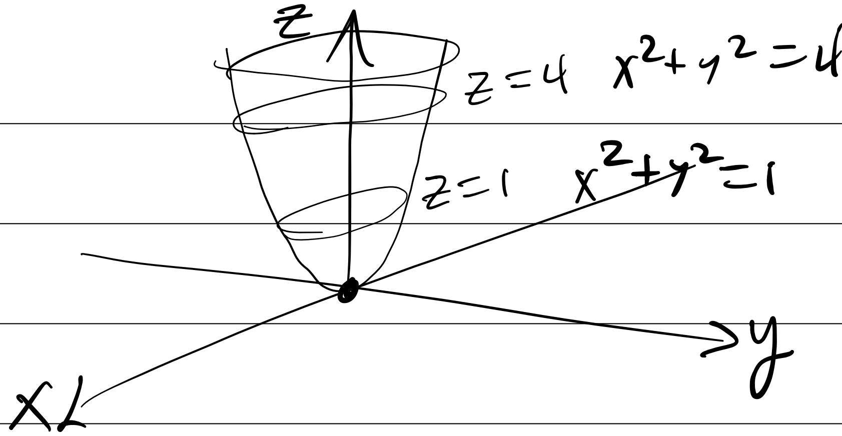
The full differential of f is

$dz = f_x\,dx + f_y\,dy$

Consider $f(x,y) = C$ as a level curve to a surface. In particular, if

$\boxed{f(x,y) = x^2+y^2}$, $\quad z = \boxed{f(x,y) = C}$

becomes $x^2+y^2=C$ (circle)

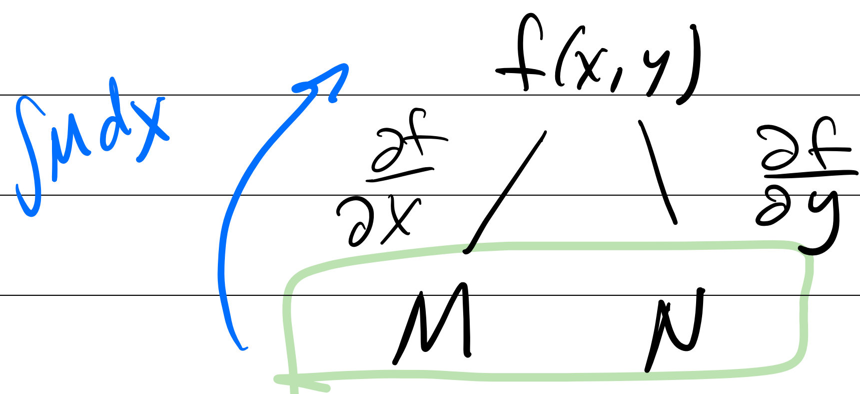


Considering the level curve $f(x,y)=C$, the (full) exact differential is

$$f_x\,dx + f_y\,dy = 0$$

For our work, we will consider $\boxed{M\,dx + N\,dy} = 0$, the solution to which is $\boxed{f(x,y)=C.}$

Basic idea:



test for exactness

$$f_{xy} = M_y \overset{?}{=} N_x = f_{yx}$$

Example: Determine whether the given di□erential equation is exact. If it is exact, solve it.

$M = f_x$

$(\sin y - y\sin x)\,dx + (\cos x + x\cos y - y)\,dy = 0$

$M = \sin y - y\sin x \qquad N = \cos x + x\cos y - y$

$M_y = \cos y - \sin x = N_x = -\sin x + \cos y$ ✓ exact

$f(x,y) = \int M\,dx = \int (\sin y - y\sin x)\,dx$

$= x\sin y + y\cos x + h(y)$

$f_y = N \Rightarrow \cancel{x\cos y} + \cancel{\cos x} + h'(y) = \cancel{\cos x} + \cancel{x\cos y} - y$

$h'(y) = -y \Rightarrow h(y) = -\frac{1}{2}y^2 + k$

We found that $f(x,y) = x\sin y + y\cos x - \frac{1}{2}y^2 + k$

Solution: $\boxed{x\sin y + y\cos x - \frac{1}{2}y^2 = C}$

check: $f_x\,dx + f_y\,dy = 0$:

$(\sin y - y\sin x)\,dx + (x\cos y + \cos x - y)\,dy = 0$

$(\sin y - y\sin x)\,dx + (\cos x + x\cos y - y)\,dy = 0$ ✓

Note: $x > 0$

Example: $\left(1 + \ln x + \frac{y}{x}\right)\, dx = (1 - \ln x)\, dy$

Form: $M\,dx + N\,dy = 0$

$(1 + \ln x + \frac{y}{x})dx - (1 - \ln x)dy = 0$

$M_y = \frac{1}{x} \qquad N_y = \frac{1}{x}$ ✓

$f(x,y) = \int M\,dx = \int (1 + \underline{\ln x} + \frac{y}{x})\,dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$f(x,y) = \cancel{x} + \underline{x\ln x - \cancel{x}} + y\ln x + h(y)$

$f_y = N \Rightarrow \ln x + h'(y) = \ln x - 1$

$h'(y) = -1 \Rightarrow h(y) = -y + k$

$$\boxed{x\ln x + y\ln x - y = C}$$

Example: Solve the given initial-value problem.

$(e^x + y)\ dx = (2 + x + ye^y)\ dy = 0 \qquad y(0) = 1$

Example: Solve the given di□erential equation by finding an appropriate integrating factor.

$$\cos x\, dx + \left(1 + \frac{2}{y}\right)\sin x\, dy = 0$$

$M = \cos x$ $\qquad$ $N = \left(1 + \frac{2}{y}\right)\sin x$

$M_y = 0$ $\qquad$ $N_x = \left(1 + \frac{2}{y}\right)\cos x \neq 0$

The integrating factor is either

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} \quad \text{OR} \quad \mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

Here, $\mu(x) = e^{\int -\frac{\cancel{(1+\frac{2}{y})}\cos x}{\cancel{(1+2/y)}\sin x} dx} = e^{-\int \frac{\cos x}{\sin x} dx}$

$$\mu = e^{-\ln|\sin x|} = \csc x$$

Then $M \csc x\, dx + N \csc x\, dy = 0$ is exact. Proceed as above.